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THE ZONE METHOD: EXPLICIT MATRIX RELATIONS FOR TOTAL EXCHANGE AREAS

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Abstract—Explicit matrix formulae are derived for the calculation of total exchange areas in the context of Hottel's zone method. Working relations are obtained for the case of a uniform grey absorbing–emitting/isotropically-scattering medium confined in a Lambert enclosure. The approach readily leads to limiting cases and significantly reduces computational labor. For an enclosure zoned into n volume and r non-black surface zones the general procedure requires evaluation of one $(n \times n)$ and one $(r \times r)$ inverse matrix. Sufficient conditions for the existence of the latter are shown to be wholly non-restrictive.

NOMENCLATURE

A_i, area of ith surface zone [m^2];
E_i, $= \sigma n^2 T_i^4$, hemispherical, black-body, emissive power of the ith surface zone [W/m^2];
$E_{g,i}$, $= \sigma n^2 T_{g,i}^4$, black-body emissive power of ith volume zone [W/m^2];
H_i, incident flux density on the ith surface zone [W/m^2];
$H_{g,i}$, incident flux density on ith volume zone [W/m^2];
K_a, absorption coefficient [m^{-1}];
K_s, scatter coefficient [m^{-1}];
K_t, $= K_a + K_s$, total extinction coefficient [m^{-1}];
q_i, $= Q_i/A_i$, net radiative flux density leaving ith surface zone [W/m^2];
Q_i, net radiative flux leaving ith surface zone [W];
S_i, net radiative absorption at ith volume zone [W];
S_i', $= S_i/V_i$, net radiative absorption per unit volume for ith volume zone [W/m^3];
$\overline{s_i s_j}$, direct surface–surface exchange area [m^2];
$\overline{s_i g_j}$, direct volume-to-surface exchange area [m^2];
$\overline{g_i s_j}$, $= \overline{s_j g_i}$, direct surface-to-volume exchange area [m^2];
$\overline{g_i g_j}$, direct volume–volume exchange area [m^2];
$\overline{S_i S_j}$, total surface–surface exchange area [m^2];
$\overline{S_i G_j}$, total volume-to-surface exchange area [m^2];
$\overline{G_i S_j}$, $= \overline{S_j G_i}$, total surface-to-volume exchange area [m^2];
$\overline{G_i G_j}$, total volume–volume exchange area [m^2];
V_i, volume of ith volume zone [m^3];
W_i, leaving flux density at ith surface zone [W/m^2];
$W_{g,i}$, leaving flux density at ith volume zone [W/m^2].

Greek symbols

ε_i, hemispherical emissivity of ith surface zone;
ρ_i, $= 1 - \varepsilon_i$, diffuse reflectivity of ith surface zone;
σ, Stefan–Boltzmann constant;
ω_0, $= K_s/K_t$, Albedo for scatter.

Matrix notation

$\mathbf{A}$, $= [A_{ij}]$;
$\mathbf{A}^T$, transpose of $\mathbf{A}$;
$\mathbf{A}^{-1}$, inverse of $\mathbf{A}$;
$\mathbf{I}$, $= [\delta_{ij}]$, the identity matrix, where δ_{ij} is the Kronecker delta;
$\mathbf{AI}$, $= [A_i \delta_{ij}]$, diagonal matrix with elements A_i;
$\mathbf{ABI}$, $= \mathbf{AI} \cdot \mathbf{BI} = [A_i B_i \delta_{ij}]$, product of diagonal matrices;
$(\mathbf{A/B})\mathbf{I}$, $\equiv \mathbf{AI} \cdot \mathbf{BI}^{-1} = [A_i/B_i \delta_{ij}]$;
$\mathbf{1}_p$, the p-vector with unity elements;
$\mathbf{K}, \mathbf{L}$, auxiliary matrices defined by equations (11) [m^2];
$\mathbf{P}$, $(n \times n)$ inverse matrix arising from (particle) scatter [m^{-2}];
$\mathbf{R}$, $(m \times m)$ inverse multiple reflection matrix [m^{-2}].

Subscripts

b, number of black surface zones;
m, $= b + r$, total number of surface zones;
n, number of volume zones;
r, number of non-black surface zones.

Superscripts

$*$, denotes dimensionless exchange area.

INTRODUCTION

THE ZONE method of analysis, due to Hottel and co-workers [2–4], is an extraordinarily versatile numerical tool for virtually any heat-transfer problem in which thermal radiation is a significant mode of energy transfer. Applications can range from simple well-mixed (one-zone) furnace models to more sophisticated multi-zone furnace models in which allowance is made for spatial dependence of gas temperature [2, 4]. Hottel and Sarofim have also shown that the method can be readily extended to the case where an absorbing–emitting medium, in addition, scatters radiation isotropically [3, 4].

Recently the essential features of the approach have been summarized in *Radiative Transfer* (R.T.) [4]. Central to the method is the definition of direct and total exchange areas for a grey gas. These grey-gas exchange areas are temperature-independent and represent basic ingredients in the formulation of real-gas spectral models [4]. Physically, in the case of an enclosure, the direct exchange areas represent direct radiant exchange between black surface zones and/or non-scattering volume zones. Alternatively, the total exchange areas also account for additional exchange between zone pairs arising from multiple reflections from the walls and/or isotropic scatter from intervening volume elements. The total exchange areas are *explicit* functions of the direct exchange areas, albedo of the medium, and the emissivity profile of the enclosure.

The procedure outlined in R.T. for the calculation of total exchange areas is non-explicit. It suggests that solution of simultaneous linear equations be performed for intermediate quantities termed "(partial)—leaving flux densities—each zone in turn, being considered the sole net emitter". Allowance for the presence of black surface zones is discussed in the context of surface–surface exchange with a transparent medium. The general working relations, however, for an absorbing–emitting and scattering medium are restricted to a grey enclosure. Moreover, because a (non-conventional) scalar notation is used to describe what are essentially matrix operations it can prove quite difficult to incorporate the suggested procedure into an efficient machine code.

The major purpose of the present work is to document certain explicit matrix elements for the calculation of total exchange areas. A general solution is thus obtained for an absorbing–emitting-isotropically scattering medium confined in a Lambert enclosure. As might be expected, in the first instance, a considerable compaction and consolidation of the scalar notation in R.T. is effected. Subsequent exploitation of matrix properties, however, leads to significant coding and computational simplifications. These should prove to be of value in practical furnace design calculations. From another viewpoint, the present work stands as an independent mathematical proof of the development in R.T. The matrix approach thus exhibits commensurate pedagogical value.

Mathematically, the zone method is perhaps the simplest numerical quadrature of the governing integral equations for radiative transfer. In conjunction with conventional finite-difference techniques to represent conductive and/or convective transport it provides a most convenient way to discretize the contribution of radiation in the general integro-differential energy equation [5]. Here, the zone size will usually be dictated by the conductive or convective length scale. Such problems may then typically give rise to extremely large arrays of exchange areas. The present work was originally motivated to address such practical difficulties as arise relative to associated efficient machine computation and represents but a portion of a broader investigation of the numerical analysis of the zone method. It is to be emphasized that the present issue is subordinate to that arising from subsequent formulation of energy balances (grey or non-grey) based on total exchange areas. The latter leads to questions of efficient solution of large sets of non-linear algebraic equations and computer store which merit separate discussion.

MATRIX PROBLEM STATEMENT

Consider an enclosure of arbitrary geometrical complexity having diffusely reflecting walls. The enclosure confines an optically uniform grey absorbing–emitting medium which, in addition, scatters radiation isotropically. Let the enclosure be subdivided into m surface zones and n volume zones each small enough to be considered isothermal. Then in matrix notation the incident and leaving hemispherical flux densities at each of the m surface zones are given by

$$\mathbf{AI} \cdot \mathbf{H} = \overline{\mathbf{ss}} \cdot \mathbf{W} + \overline{\mathbf{sg}} \cdot \mathbf{W}_g \tag{1a}$$

$$\mathbf{W} = \varepsilon\mathbf{I} \cdot \mathbf{E} + \rho\mathbf{I} \cdot \mathbf{H} \tag{1b}$$

while the net radiative flux leaving each surface zone is defined as

$$\mathbf{Q} = \mathbf{AI} \cdot (\mathbf{W} - \mathbf{H}) \tag{2a}$$

or using equation (1b) to eliminate $\mathbf{W}$

$$\mathbf{Q} = \varepsilon\mathbf{AI} \cdot (\mathbf{E} - \mathbf{H}). \tag{2b}$$

In analogy with equations (1) and (2), the incident and leaving flux densities at each of the n volume zones are given as

$$4K_t\mathbf{VI} \cdot \mathbf{H}_g = \overline{\mathbf{gs}} \cdot \mathbf{W} + \overline{\mathbf{gg}} \cdot \mathbf{W}_g \tag{3a}$$

$$\mathbf{W}_g = (1 - \omega_0)\mathbf{E}_g + \omega_0 \mathbf{H}_g \tag{3b}$$

where

$$\mathbf{S} = (1-\omega_0)4K_t\mathbf{VI}\cdot(\mathbf{H}_g-\mathbf{E}_g) \quad (4)$$

defines an n-vector whose elements represent the net radiant absorption by each volume zone, e.g. the radiation source term with respect to other modes of energy transfer.

In equations (1)–(4), **E**, **H**, **W** and **Q** are m-vectors, while $\mathbf{E}_g$, $\mathbf{H}_g$ and $\mathbf{W}_g$ are n-vectors. The matrix notation employed is essentially an obvious generalization of the scalar notation defined in R.T. For example, $\overline{\mathbf{ss}} = [\overline{s_is_j}]$ and $\overline{\mathbf{gg}} = [\overline{g_ig_j}]$ define the (symmetric) $(m \times m)$ and $(n \times n)$ arrays of direct surface–surface and volume–volume exchange areas, respectively. Further, strict adherence is made here to conventional matrix notation whereby the first and second subscripts of a scalar element define the row and column indices. It is also convenient to think of the exchange areas as being *directed* with the first subscript denoting the receiving zone and the second the sending zone. These considerations lead to the additional definition of the $(m \times n)$ array of direct volume-to-surface exchange areas, $\overline{\mathbf{sg}}$, in equation (1a) which is merely the transpose of $\overline{\mathbf{gs}}$ in equation (3a), i.e. $\overline{g_is_j} = \overline{s_jg_i}$. On this basis the integral nature of equations (1a) and (3a) become clear as they represent the flux density incident *on* a zone as a result of radiation *leaving* all other zones.

In equations (1)–(4), the notation $\mathbf{DI} = [D_i\delta_{ij}]$ is is used for all diagonal matrices. Also in what follows the simple relations for the inverse of and products involving diagonal matrices will be used freely without further comment. Lastly, all the direct exchange areas involved in equations (1)–(4) are evaluated at K_t, the total extinction coefficient.

The (arrays of) total exchange areas are defined by the following

$$\mathbf{Q} = \varepsilon\mathbf{AI}\cdot\mathbf{E} - \overline{\mathbf{SS}}\cdot\mathbf{E} - \overline{\mathbf{SG}}\cdot\mathbf{E}_g \quad (5)$$

$$\mathbf{S} = \overline{\mathbf{GG}}\cdot\mathbf{E}_g + \overline{\mathbf{GS}}\cdot\mathbf{E} - (1-\omega_0)4K_t\mathbf{VI}\cdot\mathbf{E}_g. \quad (6)$$

Consideration of thermodynamic equilibrium requires that the total exchange areas defined in equations (5) and (6) satisfy the conservation (row sum) relations

$$\varepsilon\mathbf{AI}\cdot\mathbf{l}_m = \overline{\mathbf{SS}}\cdot\mathbf{l}_m + \overline{\mathbf{SG}}\cdot\mathbf{l}_n \quad (7a)$$

$$(1-\omega_0)4K_t\mathbf{VI}\cdot\mathbf{l}_n = \overline{\mathbf{GG}}\cdot\mathbf{l}_n + \overline{\mathbf{GS}}\cdot\mathbf{l}_n. \quad (7b)$$

If equations (5) and (6) are applied to a black enclosure, confining a non-scattering medium, it is apparent that the *direct* exchange areas must satisfy

$$\mathbf{AI}\cdot\mathbf{l}_m = \overline{\mathbf{ss}}\cdot\mathbf{l}_m + \overline{\mathbf{sg}}\cdot\mathbf{l}_n \quad (8a)$$

$$4K_t\mathbf{VI}\cdot\mathbf{l}_n = \overline{\mathbf{gg}}\cdot\mathbf{l}_n + \overline{\mathbf{gs}}\cdot\mathbf{l}_m. \quad (8b)$$

Equations (8) also follow immediately from the (integral) definition of the direct exchange areas [4, 5]. Lastly, both equations (7) and (8) are useful for checking of calculations and/or generation of some exchange areas by arithmetic difference.

Equations (1)–(8) are, in fact, valid for monochromatic, as well as, grey transfer. They are however restricted by the assumptions of diffuse reflection and isotropic scatter. Both the direct and total exchange areas are independent of temperature. The total exchange areas are moreover explicitly dependent on albedo, wall emissivities, and the direct exchange areas.

GENERAL ELIMINATION PROCEDURE

The required explicit working relations for the total exchange areas are obtained by recognizing that equations (1) and (3) essentially constitute four matrix equations in the four unknowns **W**, **H**, $\mathbf{W}_g$ and $\mathbf{H}_g$. Solution for these quantities in terms of **E** and $\mathbf{E}_g$, followed by substitution of **H** and $\mathbf{H}_g$ into equations (2b) and (4) and comparison of the result with equations (5) and (6), leads to the desired result. This elimination procedure can be performed by several routes. It appears that the path now described leads to the most computationally efficient final result.

First substitute **W** and $\mathbf{W}_g$ from equations (1b) and (3b) into equations (1a) and (3a). The result is two equations in the unknowns **H** and $\mathbf{H}_g$. Simultaneous solution of these two (intermediate) equations then yields

$$\mathbf{H} = \mathbf{R}\cdot(\overline{\mathbf{ss}}+\omega_0\mathbf{L})\cdot\varepsilon\mathbf{I}\cdot\mathbf{E} + (1-\omega_0)\mathbf{R}\cdot\mathbf{K}\cdot\mathbf{E}_g \quad (9a)$$

and

$$\mathbf{H}_g = \mathbf{P}\cdot\overline{\mathbf{gs}}\cdot\mathbf{R}^T\cdot\varepsilon\mathbf{AI}\cdot\mathbf{E} + (1-\omega_0)\mathbf{P}\cdot(\overline{\mathbf{gg}}+\overline{\mathbf{gs}}\cdot\rho\mathbf{I}\cdot\mathbf{R}\cdot\mathbf{K})\cdot\mathbf{E}_g \quad (9b)$$

where **P** and **R** are inverse matrices defined as

$$\mathbf{P} = [4K_t\mathbf{VI} - \omega_0\overline{\mathbf{gg}}]^{-1} \quad (n \times n) \quad (10a)$$

$$\mathbf{R} = [\mathbf{AI} - (\overline{\mathbf{ss}}+\omega_0\mathbf{L})\cdot\rho\mathbf{I}]^{-1} \quad (m \times m) \quad (10b)$$

and **L** and **K** are auxiliaries defined by

$$\mathbf{L} = \overline{\mathbf{sg}}\cdot\mathbf{P}\cdot\overline{\mathbf{gs}} \quad (m \times m) \quad (11a)$$

$$\mathbf{K} = 4K_t\overline{\mathbf{sg}}\cdot\mathbf{P}\cdot\mathbf{VI} \quad (m \times n). \quad (11b)$$

Matrices **P** and **L** are both symmetric. To obtain equations (9) the following identities must be recognized

$$\mathbf{R}^T\cdot\mathbf{AI} = \mathbf{I} + \rho\mathbf{I}\cdot\mathbf{R}\cdot(\overline{\mathbf{ss}}+\omega_0\mathbf{L}) \quad (12a)$$

$$4K_t\mathbf{P}\cdot\mathbf{VI} = \mathbf{I} + \omega_0\mathbf{P}\cdot\overline{\mathbf{gg}} \quad (12b)$$

as well as the symmetry of the product $\rho\mathbf{I}\cdot\mathbf{R}$.

Substitution of equations (9) into equations (2b) and (4) followed by comparison with equations (5) and (6) then leads to the identities:

General solution

$$\overline{\mathbf{SS}} = \varepsilon\mathbf{AI}\cdot\mathbf{R}\ (\overline{\mathbf{ss}}+\omega_0\mathbf{L})\cdot\varepsilon\mathbf{I} \quad (m \times m) \quad (13a)$$

$$\overline{\mathbf{SG}} = (1-\omega_0)\varepsilon\mathbf{AI}\cdot\mathbf{R}\cdot\mathbf{K} = \overline{\mathbf{GS}}^T \quad (m \times n) \quad (13b, c)$$

$$\overline{\mathbf{GG}} = (1-\omega_0)^2 4K_t\mathbf{VI}\cdot\mathbf{P}\cdot\overline{\mathbf{gg}} + (1-\omega_0)^2\mathbf{K}^T\cdot\rho\mathbf{I}\cdot\mathbf{R}\cdot\mathbf{K} \quad (n \times n). \quad (13d)$$

Equations (13) are the desired final result. They are valid for all $0 \leqslant \omega_0 \leqslant 1$ and $0 \leqslant \varepsilon_i \leqslant 1$. Note further that equations (13) imply the symmetry of the groupings $\mathbf{AI}\cdot\mathbf{R}\cdot(\overline{\mathbf{ss}}+\omega_0\mathbf{L})$ and $\mathbf{VI}\cdot\mathbf{P}\cdot\overline{\mathbf{gg}}$ which can be used to computational advantage. All such symmetry relations can, in fact, be easily proved provided that the inverse matrices **P** and **R** exist. Sufficient conditions for the latter are considered in the Appendix. These are shown to be wholly non-restrictive for any physically meaningful enclosure problem

LIMITING FORMS

If $\omega_0 = 0$, $\mathbf{P} = (1/4K_t)\mathbf{VI}^{-1}$ and $\mathbf{K} = \overline{\mathbf{sg}}$; while if $\varepsilon_i = 1$, $\mathbf{R} = \mathbf{AI}^{-1}$. Thus, if $\omega_0 = 0$ and all $\varepsilon_i = 1$, equations (13) reduce identically to expressions for the direct exchange areas. More important limiting forms resulting from equations (13) are documented as follows:

No scatter ($\omega_0 = 0$)

$$\overline{\mathbf{SS}} = \varepsilon\mathbf{AI}\cdot\mathbf{R}\cdot\overline{\mathbf{ss}}\cdot\varepsilon\mathbf{I} \quad (14a)$$

$$\overline{\mathbf{SG}} = \varepsilon\mathbf{AI}\cdot\mathbf{R}\cdot\overline{\mathbf{sg}} = \overline{\mathbf{GS}}^T \quad (14b, c)$$

$$\overline{\mathbf{GG}} = \overline{\mathbf{gg}}+\overline{\mathbf{gs}}\cdot\rho\mathbf{I}\cdot\mathbf{R}\cdot\overline{\mathbf{sg}}. \quad (14d)$$

Perfect isotropic scatter ($\omega_0 = 1$)

$$\overline{\mathbf{SS}} = \varepsilon\mathbf{AI}\cdot\mathbf{R}\cdot(\overline{\mathbf{ss}}+\mathbf{L})\cdot\varepsilon\mathbf{I} \quad (15a)$$

$$\overline{\mathbf{SG}} = \mathbf{0} = \overline{\mathbf{GS}}^T \quad (15b, c)$$

$$\overline{\mathbf{GG}} = \mathbf{0}. \quad (15d)$$

Black enclosure (all $\varepsilon_i = 1$)

$$\overline{\mathbf{SS}} = \overline{\mathbf{ss}}+\omega_0\mathbf{L} \quad (16a)$$

$$\overline{\mathbf{SG}} = (1-\omega_0)\mathbf{K} = \overline{\mathbf{GS}}^T \quad (16b, c)$$

$$\overline{\mathbf{GG}} = (1-\omega_0)^2 4K_t\mathbf{VI}\cdot\mathbf{P}\cdot\overline{\mathbf{gg}}. \quad (16d)$$

Both the fundamental physical and mathematical significance of the auxiliary matrices **K** and **L** are made clear from equations (16).

Optically thin/transparent approximation

Here use is made of the fact that in the limit as $K_t \to 0$

$$\overline{\mathbf{ss}} \to \text{finite}, \qquad \overline{\mathbf{sg}} \to \mathbf{0}$$

$$\frac{1}{K_t}\overline{\mathbf{gs}} \to \text{finite}, \qquad \frac{1}{K_t}\overline{\mathbf{gg}} \to \mathbf{0}$$

which results in $\mathbf{P} \to (1/4K_t)\mathbf{VI}^{-1}$, $\mathbf{L} \to \mathbf{0}$, and $\mathbf{K} \to \mathbf{0}$. Equations (13) then become as $\lim K_t \to 0$

$$\overline{\mathbf{SS}} \to \varepsilon\mathbf{AI}\cdot\mathbf{R}\cdot\overline{\mathbf{ss}}\cdot\varepsilon\mathbf{I} \quad (17a)$$

$$\overline{\mathbf{SG}} \to \mathbf{0} \quad (17b)$$

$$\frac{1}{K_t}\overline{\mathbf{GS}} \to (1-\omega_0)\left[\frac{1}{K_t}\overline{\mathbf{gs}}\right]\cdot\mathbf{R}^T\cdot\varepsilon\mathbf{AI} \quad (17c)$$

$$\frac{1}{K_t}\overline{\mathbf{GG}} \to \mathbf{0}. \quad (17d)$$

The total exchange areas as computed from equations (17) are then used in conjunction with the following simplified forms of equations (5) and (6), viz.

$$\mathbf{Q} = \varepsilon\mathbf{AI}\cdot\mathbf{E}-\overline{\mathbf{SS}}\cdot\mathbf{E} \quad (18)$$

$$\frac{1}{K_t}\mathbf{S} = \frac{1}{K_t}\overline{\mathbf{GS}}\cdot\mathbf{E}-(1-\omega_0)4\mathbf{VI}\cdot\mathbf{E}_g. \quad (19)$$

In this instance transfer between the walls and transfer from walls to volume zones assumes the intervening medium to be transparent.

MATRIX PARTITIONING

It is clear from equations (10) and (13) that the general formulation requires only two inverse matrices to be calculated. Matrix **P** is symmetric and positive for $K_t > 0$ and $\omega_0 \neq 0$ (see Appendix). When some of the surface zones are black, however, matrix $\mathbf{R}^{-1}$ can be reduced so as to obtain simplifications in numerical labor.

Let the surface zones be numbered such that the first b are black and the remaining r are non-black ($m = r+b$). In what follows the partition notation as in

$$\underset{(m\times m)}{\overline{\mathbf{ss}}} = \begin{bmatrix} \overset{(b\times b)}{\overline{\mathbf{ss}}_{1,1}} & \overset{(b\times r)}{\overline{\mathbf{ss}}_{1,2}} \\ \underset{(r\times b)}{\overline{\mathbf{ss}}_{2,1}} & \underset{(r\times r)}{\overline{\mathbf{ss}}_{2,2}} \end{bmatrix} \quad (20a)$$

shall be adopted for all $(m \times m)$ arrays as required. Further, partition $\overline{\mathbf{sg}}$ and $\overline{\mathbf{gs}}$ as

$$\underset{(m\times n)}{\overline{\mathbf{sg}}} = \begin{bmatrix} \overset{(b\times n)}{\overline{\mathbf{sg}}_1} \\ \underset{(r\times n)}{\overline{\mathbf{sg}}_2} \end{bmatrix} \qquad \underset{(n\times m)}{\overline{\mathbf{gs}}} = [\, \overset{(n\times b)}{\overline{\mathbf{gs}}_1} \;\vdots\; \overset{(n\times r)}{\overline{\mathbf{gs}}_2} \,] \quad (20b, c)$$

such that the auxiliary matrix **L** partitions as

$$\mathbf{L} = \begin{bmatrix} \overline{\mathbf{sg}}_1\cdot\mathbf{P}\cdot\overline{\mathbf{gs}}_1 & \overline{\mathbf{sg}}_1\cdot\mathbf{P}\cdot\overline{\mathbf{gs}}_2 \\ \overline{\mathbf{sg}}_2\cdot\mathbf{P}\cdot\overline{\mathbf{gs}}_1 & \overline{\mathbf{sg}}_2\cdot\mathbf{P}\cdot\overline{\mathbf{gs}}_2 \end{bmatrix}. \quad (20d)$$

With the surface zone numbering convention thus adopted, it follows that matrix $\mathbf{R}^{-1}$ and thus inverse **R** are both upper (block) triangular. The partitions of **R** are then computed readily from

$$\mathbf{R}_{1,1} = \mathbf{AI}_{1,1}^{-1}; \qquad \mathbf{R}_{1,2} = \mathbf{AI}_{1,1}^{-1}\cdot[\overline{\mathbf{ss}}_{1,2}+\omega_0\mathbf{L}_{1,2}]\cdot\mathbf{T} \quad (21a, b)$$

$$\mathbf{R}_{2,1} = \mathbf{0}\ (r\times b); \quad \mathbf{R}_{2,2} = \rho\mathbf{I}_{2,2}^{-1}\cdot\mathbf{T} \quad (21c, d)$$

where

$$\mathbf{T} = [(\mathbf{A}/\rho)\mathbf{I}_{2,2}-(\overline{\mathbf{ss}}_{2,2}+\omega_0\mathbf{L}_{2,2})]^{-1} \quad (22)$$

is a symmetric $(r \times r)$ inverse matrix. Matrix **T** is a generalization of the "transfer" matrix defined in R.T.

Matrix partitioning and symmetry relations may be used at length to simplify the matrix multiplications incurred by equations (13). The Appendix discusses further simplifications in the calculation of matrix **T** effected by appropriate numbering of the *grey* surface zones. When an enclosure exhibits geometric symmetry, partitioning based on appropriate zone numbering will also expedite evaluation of the inverse matrices. Such conventions will usually take precedence over that for black surface zones.

THE LIMIT OF SMALL ZONE SIZE

As the zone size decreases, all the direct and total exchange areas vanish. Further, in systems involving one or two-dimensional geometries, the zones will have at least one infinite dimension. Strictly speaking then the arrays **VI** and **AI** have no meaning here. Both of these situations can be addressed through the definition of dimensionless exchange areas based on a unit volume or area of the receiving zone.

Let dimensionless direct exchange areas be defined as follows:

$$\left.\begin{aligned} \overline{\mathbf{ss}}^* &= \mathbf{AI}^{-1}\cdot\overline{\mathbf{ss}}; \quad & \overline{\mathbf{sg}}^* &= \mathbf{AI}^{-1}\cdot\overline{\mathbf{sg}} \\ \overline{\mathbf{gs}}^* &= \frac{1}{K_t}\mathbf{VI}^{-1}\cdot\overline{\mathbf{gs}}; \quad & \overline{\mathbf{gg}}^* &= \frac{1}{K_t}\mathbf{VI}^{-1}\cdot\overline{\mathbf{gg}} \end{aligned}\right\} \quad (23)$$

i.e.

$$\overline{\mathbf{ss}}^* = [\overline{s_i s_j}/A_i], \quad \overline{\mathbf{gs}}^* = \left[\frac{1}{K_t}\overline{g_i s_j}/V_i\right],$$

etc. Similar notation is used for arrays of dimensionless total exchange areas such that pre-multiplication of equations (5) and (6) by $\mathbf{AI}^{-1}$ and $(1/K_t)\mathbf{VI}^{-1}$, respectively, yields

$$\mathbf{q} = \varepsilon\mathbf{I}\cdot\mathbf{E} - \overline{\mathbf{SS}}^*\cdot\mathbf{E} - \overline{\mathbf{SG}}^*\cdot\mathbf{E}_g \quad (24)$$

$$\frac{1}{K_t}\mathbf{S}' = \overline{\mathbf{GG}}^*\cdot\mathbf{E}_g + \overline{\mathbf{GS}}^*\cdot\mathbf{E} - 4(1-\omega_0)\ \mathbf{E}_g. \quad (25)$$

Here $\mathbf{q} = [Q_i/A_i]$ is the m-vector of flux densities and $\mathbf{S}' = [S_i/V_i]$ is the (dimensional) radiant source per unit volume.

Pre-multiplication of equations (13a, b) by $\mathbf{AI}^{-1}$ and (13c, d) by $(1/K_t)\mathbf{VI}^{-1}$, respectively, similarly yields

$$\overline{\mathbf{SS}}^* = \varepsilon\mathbf{I}\cdot\mathbf{R}^*\cdot\left(\overline{\mathbf{ss}}^* + \frac{\omega_0}{4}\mathbf{L}^*\right)\cdot\varepsilon\mathbf{I} \quad (26a)$$

$$\overline{\mathbf{SG}}^* = (1-\omega_0)\varepsilon\mathbf{I}\cdot\mathbf{R}^*\ \overline{\mathbf{sg}}^*\cdot\mathbf{P}^* \quad (26b)$$

$$\overline{\mathbf{GS}}^* = (1-\omega_0)\mathbf{P}^*\cdot\overline{\mathbf{gs}}^*\cdot\mathbf{R}^{*\prime}\cdot\varepsilon\mathbf{I} \quad (26c)$$

$$\overline{\mathbf{GG}}^* = (1-\omega_0)^2\mathbf{P}^*\cdot\overline{\mathbf{gg}}^* + (1-\omega_0)^2\mathbf{P}^*\cdot\overline{\mathbf{gs}}^*\cdot\rho\mathbf{I}\cdot\mathbf{R}^*\cdot\overline{\mathbf{sg}}^*\cdot\mathbf{P}^* \quad (26d)$$

where

$$\mathbf{P}^* \equiv \left[\mathbf{I} - \frac{\omega_0}{4}\overline{\mathbf{gg}}^*\right]^{-1} \quad (27a)$$

$$\mathbf{L}^* \equiv \overline{\mathbf{sg}}^*\cdot\mathbf{P}^*\cdot\overline{\mathbf{gs}}^* \quad (27b)$$

$$\mathbf{R}^* \equiv \left[\mathbf{I} - \left(\overline{\mathbf{ss}}^* + \frac{\omega_0}{4}\mathbf{L}^*\right)\cdot\rho\mathbf{I}\right]^{-1} \quad (27c)$$

$$\mathbf{R}^{*\prime} \equiv \left[\mathbf{I} - \rho\mathbf{I}\cdot\left(\overline{\mathbf{ss}}^* + \frac{\omega_0}{4}\mathbf{L}^*\right)\right]^{-1} \quad (27d)$$

such that

$$\mathbf{R}^{*\prime} = \mathbf{AI}^{-1}\cdot\mathbf{R}^{*T}\cdot\mathbf{AI}. \quad (28)$$

Likewise, the conservation relations for the dimensionless total exchange areas are found from equations (7) to be

$$\varepsilon\mathbf{I}\cdot\mathbf{I}_m = \overline{\mathbf{SS}}^*\cdot\mathbf{I}_m + \overline{\mathbf{SG}}^*\cdot\mathbf{I}_n \quad (29a)$$

$$4(1-\omega_0)\mathbf{I}_n = \overline{\mathbf{GG}}^*\cdot\mathbf{I}_n + \overline{\mathbf{GS}}^*\cdot\mathbf{I}_m. \quad (29b)$$

With this formulation, when the zone areas and volumes are unequal, none of the symmetry relations derived for the dimensional exchange areas are valid. In addition, it follows that

$$\overline{\mathbf{GS}}^* = \frac{1}{K_t}\mathbf{VI}^{-1}\cdot\overline{\mathbf{SG}}^{*T}\cdot\mathbf{AI} \quad (30)$$

so that $\overline{\mathbf{GS}}^* \neq \overline{\mathbf{SG}}^{*T}$ even if the zone areas and volumes *are* equal. Still, only two inverse matrices need be calculated despite the appearance of $\mathbf{R}^{*\prime}$ in equation (26c). Here matrix $\mathbf{R}^{*\prime}$ can be partitioned into (lower) block triangular form when black zones are present and the identity

$$\rho\mathbf{I}\cdot\mathbf{R}^* = \mathbf{R}^{*\prime}\cdot\rho\mathbf{I} \quad (31a)$$

leads to

$$\mathbf{R}^{*\prime}_{2,2} = \rho\mathbf{I}_{2,2}\cdot\mathbf{R}^*_{2,2}\cdot\rho\mathbf{I}^{-1}_{2,2}. \quad (31b)$$

It should be clear that the major additional labor posed by this approach is that attendant to non-symmetric multiplication and matrix inversion. For the special case of $V_i =$ const., $A_i =$ const. all the (dimensional) symmetry relations hold with the particular consequences that $\mathbf{R}^{*\prime} = \mathbf{R}^{*T}$ and $\overline{\mathbf{SG}}^*$ is a constant multiplicative of $\overline{\mathbf{GS}}^{*T}$.

In the limit of arbitrarily small zone size, the dimensionless exchange areas represent "point-point" exchange and do not vanish. Moreover, equations (24)–(28) are the form to be used when the radiation balances are struck at infinitesimal elements at the centers of each zone, i.e. "finite–infinitesimal zone exchange". This is the approach promulgated by Einstein [1] and elaborated upon by Noble [5]. In this regard, the dimensionless exchange areas are to be interpreted as derivatives, viz.

$$\overline{\mathbf{ss}}^* = \left[\frac{\partial(\overline{s_i s_j})}{\partial A_i}\right], \quad \overline{\mathbf{gs}}^* = \left[\frac{1}{K_t}\frac{\partial(\overline{g_i s_j})}{\partial V_i}\right], \quad \text{etc.}$$

EXAMPLE PROBLEMS

The simplicity and utility of the matrix approach are evidenced through three examples.

Example I

Two infinite parallel plates, with emissivities ε_1 and ε_2, confine a transparent medium of width L. Evaluate all the surface–surface total exchange areas.

Solution. Application of equation (26a) with $\omega_0 = 0$ yields

$$\overline{\mathbf{SS}}^* = \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & -\rho_2 \\ -\rho_1 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}$$

$$= 1/(1-\rho_1\rho_2) \begin{bmatrix} \varepsilon_1^2\rho_2 & \varepsilon_1\varepsilon_2 \\ \varepsilon_2\varepsilon_1 & \varepsilon_2^2\rho_1 \end{bmatrix}.$$

Example II

An enclosure of arbitrary geometry confines a non-scattering medium having absorption coefficient $K_a = K_t$. The surface of the enclosure is divided into b black and r non-black surface zones ($m = r+b$). None of the non-black zones have a direct view of each other (they may, for example, comprise one planar wall). The enclosure is to be divided into n volume zones. Find closed-form expressions for the total volume–volume exchange areas.

Solution. The dimensionless form of equation (14d) always partitions as

$$\overline{\mathbf{GG}}^* = \overline{\mathbf{gg}}^* + \overline{\mathbf{gs}}_2^* \cdot \rho\mathbf{I}_{2,2} \cdot \mathbf{R}_{2,2}^* \cdot \overline{\mathbf{sg}}_2^* \qquad (32)$$

and since $\overline{\mathbf{ss}}_{2,2}^* = \mathbf{0}$, it follows from equation (21d) and (22) that $\mathbf{R}_{2,2}^* = \mathbf{I}_{2,2}$. The scalar form of equation (32) is then simply

$$\overline{G_iG_j^*} = \overline{g_ig_j^*} + \sum_{k=b+1}^{m} \overline{g_is_k^*} \cdot \rho_k \cdot \overline{s_kg_j^*}.$$

Example III

Repeat Example I for the case of a purely isotropic scattering medium. The optical thickness between the plates is $\tau_0 = K_tL = 1$, the emissivities are $\varepsilon_1 = 0{\cdot}6$ and $\varepsilon_2 = 0{\cdot}4$, and $n = 3$ equi-width volume zones are to be used.

Solution. In this case the required direct exchange areas can be computed from forms involving the third-order exponential integral as documented in Hottel and Sarofim [4]. Numerically, for $\tau_0 = 1$, these are found to be

$$\overline{\mathbf{ss}}^* = \begin{bmatrix} 0 & 0{\cdot}219 \\ 0{\cdot}219 & 0 \end{bmatrix} \quad \overline{\mathbf{sg}}^* = \begin{bmatrix} 0{\cdot}430 & 0{\cdot}222 & 0{\cdot}129 \\ 0{\cdot}129 & 0{\cdot}222 & 0{\cdot}430 \end{bmatrix}$$

$$\overline{\mathbf{gs}}^* = \begin{bmatrix} 1{\cdot}291 & 0{\cdot}386 \\ 0{\cdot}665 & 0{\cdot}665 \\ 0{\cdot}386 & 1{\cdot}291 \end{bmatrix} \quad \overline{\mathbf{gg}}^* = \begin{bmatrix} 1{\cdot}419 & 0{\cdot}626 & 0{\cdot}278 \\ 0{\cdot}636 & 1{\cdot}419 & 0{\cdot}626 \\ 0{\cdot}278 & 0{\cdot}626 & 1{\cdot}419 \end{bmatrix}.$$

When $\rho\mathbf{I}^{-1}$ exists, equation (27c) yields the identity

$$\mathbf{R}^* \cdot \left(\overline{\mathbf{ss}}^* + \frac{\omega_0}{4}\mathbf{L}^* \right) = (\mathbf{R}^* - \mathbf{I}) \cdot \rho\mathbf{I}^{-1}, \qquad (33)$$

which, when used to simplify equation (26a) results in

$$\overline{\mathbf{SS}}^* = (\varepsilon/\rho)\mathbf{I} \cdot (\mathbf{T}^* - \rho\mathbf{I}) \cdot (\varepsilon/\rho)\mathbf{I} \qquad (34)$$

where

$$\mathbf{T}^* \equiv \left[\rho\mathbf{I}^{-1} - \left(\overline{\mathbf{ss}}^* + \frac{\omega_0}{4}\mathbf{L}^* \right) \right]^{-1}. \qquad (35)$$

For $\omega_0 = 1$, equations (27) then produce

$$\mathbf{P}^* = \begin{bmatrix} 1{\cdot}700 & 0{\cdot}485 & 0{\cdot}301 \\ 0{\cdot}485 & 1{\cdot}785 & 0{\cdot}485 \\ 0{\cdot}301 & 0{\cdot}485 & 1{\cdot}700 \end{bmatrix} \quad \mathbf{L}^* = \begin{bmatrix} 1{\cdot}752 & 1{\cdot}371 \\ 1{\cdot}371 & 1{\cdot}752 \end{bmatrix}$$

and

$$\mathbf{T}^* = \begin{bmatrix} 0{\cdot}554 & 0{\cdot}253 \\ 0{\cdot}253 & 0{\cdot}930 \end{bmatrix}$$

which, when substituted into equation (34) yields

$$\overline{\mathbf{SS}}^* = \begin{bmatrix} 0{\cdot}347 & 0{\cdot}253 \\ 0{\cdot}253 & 0{\cdot}147 \end{bmatrix}.$$

The only approximation here is that of $n = 3$ volume zones. It may be shown that

$$\lim_{n\to\infty} \mathbf{L}^* = \begin{bmatrix} 1{\cdot}786 & 1{\cdot}336 \\ 1{\cdot}336 & 1{\cdot}786 \end{bmatrix}$$

which produces $\overline{S_1S_2^*} = 0{\cdot}252$. (Note: all calculations have been rounded down from six significant figures for expedient presentation.) The error is less than 0·5 per cent and is a function of τ_0/n. For large n, calculation of $\mathbf{P}^*$ might be expedited because of geometric symmetry. Thus, if the volume zones are numbered with the sequence $1, 2, \ldots, n/2, n, n-1, \ldots, n/2+1$ (n even), $\mathbf{P}^{*-1}$ will partition into four square (symmetric) sub-matrices of dimension $n/2$. Only two of these are unique with those diagonally opposed being identical. It follows that the equivalent partitions of $\mathbf{P}^*$ exhibit the very same properties and may be obtained by inverting two matrices each of dimension $n/2$.

DISCUSSION

The relationship between the matrix approach and the scalar procedure outlined in R.T. is now examined. When written in the present matrix notation, the latter is seen to consist of two parts, viz.

$$\underset{(n\times m)\qquad\qquad (n\times n)}{\overset{(m\times m)\qquad\qquad (m\times n)}{\left[\begin{array}{c|c} -[(\mathbf{A}/\rho)\mathbf{I} - \overline{\mathbf{ss}}] & \overline{\mathbf{sg}} \\ \hline \overline{\mathbf{gs}} & -(1/\omega_0)\mathbf{P}^{-1} \end{array}\right]}} \cdot \underset{(n\times m)\ (n\times n)}{\overset{(m\times m)\ (m\times n)}{\left[\begin{array}{c|c} \mathbf{W}_p^{\circledS} & \mathbf{W}_p^{\textcircled{G}} \\ \hline \mathbf{W}_{g,p}^{\circledS} & \mathbf{W}_{g,p}^{\textcircled{G}} \end{array}\right]}} =$$

$$- \underset{(n\times n)}{\overset{(m\times m)}{\left[\begin{array}{c|c} (\varepsilon\mathbf{A}/\rho)\mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & 4K_t(1-\omega_0)/\omega_0\mathbf{VI} \end{array}\right]}} \cdot \underset{(n\times n)}{\overset{(m\times m)}{\left[\begin{array}{c|c} \mathbf{EI} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{E}_g\mathbf{I} \end{array}\right]}} \qquad (36)$$

followed by the substitutions

$$\overline{\mathbf{SS}} = (\varepsilon\mathbf{A}/\rho)\mathbf{I} \cdot [\mathbf{W}_p^{\circledS} \cdot \mathbf{EI}^{-1} - \varepsilon\mathbf{I}] \qquad (37a)$$

$$\overline{\mathbf{SG}} = (\varepsilon\mathbf{A}/\rho)\mathbf{I} \cdot \mathbf{W}_p^{\textcircled{G}} \cdot \mathbf{E}_g\mathbf{I}^{-1} \qquad (37b)$$

$$\overline{\mathbf{GG}} = \overline{\mathbf{gg}} + \overline{\mathbf{gs}} \cdot \mathbf{W}_p^{\textcircled{G}} \cdot \mathbf{E}_g\mathbf{I}^{-1} \qquad (\omega_0 = 0) \qquad (37c)$$

or if $\omega_0 \neq 0$

$$\overline{\mathbf{GS}} = 4K_t(1-\omega_0)/\omega_0 \mathbf{VI}\cdot\mathbf{W}_{g,p}^{\text{Ⓢ}}\cdot\mathbf{EI}^{-1} \quad (37d)$$

$$\overline{\mathbf{GG}} = 4K_t(1-\omega_0)/\omega_0\mathbf{VI}\cdot[\mathbf{W}_{g,p}^{\text{Ⓖ}}\cdot\mathbf{E}_g\mathbf{I}^{-1}-(1-\omega_0)\mathbf{I}]. \quad (37e)$$

In equations (36) and (37) **EI** and $\mathbf{E}_g\mathbf{I}$ are arbitrary diagonal arrays of emissive powers which cancel on substitution. Further, the second member in the l.h.s. of equation (36) is an $(m+n)^2$ array of (partitioned) "partial leaving flux densities—each zone in turn being the sole net emitter" [4]. The circled superscript notation in equation (36) refers to the emitting zones. Thus the typical element of $(m \times n)$ array $\mathbf{W}_p^{\text{Ⓖ}}$, $W_{ij}^{\text{Ⓖ}}$, for example, is the leaving flux density at surface zone i caused by sole net emission of volume zone j.

Equation (36) exemplifies the basic mathematical link between the matrix approach and that suggested in R.T., viz. if a set of linear equations is repeatedly solved putting the constant vector equal to successive columns of the identity matrix, then the solution vectors, so-obtained, are the columns of the inverse of the original coefficient matrix. The minimum computational labor implied by equation (36) is the determination of a (symmetric) inverse matrix of dimension $(m+n)$. It thus follows that evaluation of the leaving flux densities by methods of solution for simultaneous equations, as suggested in R.T. would be inappropriate. In the following comparisons it will be assumed that the inverse implied by equation (36) has been calculated as such.

As written, equation (36) applies only if $\omega_0 \neq 0$ and if the enclosure is grey, since the existence of $\boldsymbol{\rho}\mathbf{I}^{-1}$ is assumed. If equation (36) is pre-multiplied by the $(m+n)^2$ block diagonal matrix whose upper and lower elements are $\boldsymbol{\rho}\mathbf{I}$ $(m \times m)$ and $\omega_0\mathbf{I}$ $(n \times n)$, respectively, the result is valid for all ω_0 and ε_i. Performing this operation and carrying out the implied partitioning in equation (36) yields

$$\mathbf{W}_p^{\text{Ⓢ}} = \mathbf{R}^T\cdot\varepsilon\mathbf{AI}\cdot\mathbf{EI} \qquad (m \times m) \quad (38a)$$

$$\mathbf{W}_p^{\text{Ⓖ}} = (1-\omega_0)\boldsymbol{\rho}\mathbf{I}\cdot\mathbf{R}\cdot\mathbf{K}\cdot\mathbf{E}_g\mathbf{I} \qquad (m \times n) \quad (38b)$$

plus the interrelationships

$$\mathbf{W}_{g,p}^{\text{Ⓢ}} = \omega_0\mathbf{P}\cdot\overline{\mathbf{gs}}\cdot\mathbf{W}_p^{\text{Ⓢ}} \qquad (n \times m) \quad (39a)$$

$$\mathbf{W}_{g,p}^{\text{Ⓖ}} = \omega_0\mathbf{P}\cdot\overline{\mathbf{gs}}\cdot\mathbf{W}_p^{\text{Ⓖ}} + (1-\omega_0)4K_t\mathbf{P}\cdot\mathbf{VI}\cdot\mathbf{E}_g\mathbf{I} \qquad (n \times n) \quad (39b)$$

which are valid for all ω_0 and ε_i.

Equations (38) and (39) show clearly that neither $\mathbf{W}_p^{\text{Ⓢ}}$ and $\mathbf{W}_{g,p}^{\text{Ⓖ}}$ are symmetric, nor is $\mathbf{W}_p^{\text{Ⓖ}}$ the transpose of $\mathbf{W}_{g,p}^{\text{Ⓢ}}$. This can lead to a valid source of confusion regarding the scalar notation in R.T., where the row and column indices are reversed. When black surface zones are present, equations (38) will be sparse corresponding to the partitioning of **R**.

Comparison of equations (38) with equations (13) yields

$$\overline{\mathbf{SS}} = \varepsilon\mathbf{I}\cdot(\overline{\mathbf{ss}}+\omega_0\mathbf{L})\cdot\mathbf{W}_p^{\text{Ⓢ}}\cdot\mathbf{EI}^{-1} \quad (40a)$$

$$\overline{\mathbf{GS}} = (1-\omega_0)\mathbf{K}^T\cdot\mathbf{W}_p^{\text{Ⓢ}}\cdot\mathbf{EI}^{-1} \quad (40b)$$

$$\overline{\mathbf{GG}} = (1-\omega_0)4K_t\mathbf{VI}\cdot\mathbf{P}\cdot[(1-\omega_0)\overline{\mathbf{gg}}+\overline{\mathbf{gs}}\cdot\mathbf{W}_p^{\text{Ⓖ}}\cdot\mathbf{E}_g\mathbf{I}^{-1}] \quad (40c)$$

which are universally valid forms involving the leaving flux densities.

That the partitioned-$\mathbf{W}_p$ notation is surely a universally valid way to approach the problem is evident from equations (38)–(40). These equations do make it quite clear, however, that the notation is less fundamental and unnecessarily restrictive, mathematically. In this regard, note that equations (37b,d) produce identical results to equation (40b), when expanded. Equation (37a) does, however, suggest the saving of one non-sparse matrix multiplication when $\boldsymbol{\rho}\mathbf{I}^{-1}$ exists. This results when equation (12a) is substituted into equation (13a) for the product $\mathbf{R}\cdot(\overline{\mathbf{ss}}+\omega_0\mathbf{L})$. A similar saving occurs when $\omega_0 \neq 0$ and equation (12b) is substituted into equation (13d). The first device has been used in Example III and can be employed generally to simplify the calculation of $\overline{\mathbf{SS}}_{2,2}$. In no other instance do equations (36) and (37) appear to produce simplifications of equations (13).

For a non-scattering medium a major advantage of the matrix approach is a significant simplification of the notation in R.T. when black surface zones are to be allowed for. The latter does, in fact, suggest certain column operations on the "transfer matrix" which permit calculation of $\overline{\mathbf{SS}}$ in a transparent medium when black zones are present. This procedure could be generalized to an absorbing–emitting medium where it is tantamount to partitioning $\mathbf{W}_p^{\text{Ⓢ}}$ and $\mathbf{W}_p^{\text{Ⓖ}}$ (or **R**). It would replace equations (14) with nine scalar equations. Implementation of these suggestions to the case $\omega_0 \neq 0$ with a scalar notation would be virtually impossible.

For the case of $\varepsilon_i \neq 1$ and $\omega_0 = 0$ the present method results in identical work to the procedure in R.T. The major saving in numerical labor arises in the case of $\omega_0 \neq 0$ where the calculation of one $(n \times n)$ and one $(m \times m)$ inverse is required as against one $(m+n)^2$ inverse. Comparison of computational labor is difficult as marked trade-offs exist between storage, execution time and coding complexity. Notwithstanding, for $m = 40$, $n = 100$ and $\varepsilon_i \neq 1$, it is estimated that evaluation of $\overline{\mathbf{SS}}$ for $\omega_0 = 1$ by equations (36) and (37a) might require 40 per cent more machine time and about 8K of additional store as compared with the matrix procedure. This assumes labor for matrix inversion to be proportional to the cube of the dimension and accounts for the extra matrix multiplications incurred by equation (11a). A direct comparison for some $\varepsilon_i = 1$ and $\omega_0 \neq 0$ is not possible.

CONCLUSIONS

A consistent matrix approach has been presented for the explicit calculation of total exchange areas. Resultant formulae are restricted to diffuse reflection, isotropic scatter and spatially uniform optical properties. The method effects considerable computational and notational simplifications over the scalar procedure outlined in *Radiative Transfer* [4]. These are demonstrated with three examples. Because the procedure is explicit, formulation of limiting cases, notably the optically thin approximation, follow as particularly simple consequences of the more general result. Formulae are also presented for "finite-infinitesimal zone exchange".

Results obtained here are particularly suited for the development of a highly efficient subroutine for incorporation into a machine code for furnace design calculations. The procedure leads to both ready estimates of computational labor and efficient storage of large arrays. Here internal matrix subroutines may be used and symmetry relations provide a basis for checks and computational savings.

It is shown that the major computational advantage for this method over that described in *Radiative Transfer* occurs when the medium scatters. The procedure then requires calculation of two (symmetric) inverse matrices of dimensions $(n \times n)$ and $(r \times r)$ where n and r are the number of volume and non-black surface zones respectively. For a non-scattering medium, only one $(r \times r)$ inverse need be calculated. Sufficient conditions are obtained which guarantee existence of the inverse matrices for any physically meaningful enclosure problem.

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APPENDIX

Sufficient Conditions for Existence of the Inverse Matrices

By definition all enclosure problems will satisfy the following conditions:

(C1) The enclosure is a connected region.

(C2) Each surface zone will have the direct view of at least one other surface zone.

(C3) For $K_t > 0$ each surface zone will have the direct view of at least one volume zone.

With these conditions it is now shown that for $\omega_0 \neq 0$ and $K_t > 0$, matrix $\mathbf{P}$ exists and is positive. Further, $\mathbf{R}$ exists subject to the sufficient conditions:

(1) $K_t > 0$; or

(2) if $K_t = 0$ the enclosure must contain at least one black surface zone; or

(3) if $K_t = 0$ and no $\varepsilon_i = 1$, there must be one i such that $\varepsilon_i \neq 0$.

Contradiction of (3) leads to a trivial problem.

Inverse (particle) scatter matrix $\mathbf{P}$

The existence of $\mathbf{P}$ need be examined only for the case $\omega_0 \neq 0$ and $K_t > 0$. Under these conditions $\mathbf{P}^{-1}$ is irreducible since (C1) implies $\overline{\mathbf{gg}}$ to be irreducible. The off-diagonal elements of $\mathbf{P}^{-1}$ will also be non-positive. Now the n-vector of row sums, $\mathbf{\Sigma}$, of $\mathbf{P}^{-1}$ is defined by

$$\mathbf{\Sigma} = [4K_t\mathbf{VI} - \omega_0\overline{\mathbf{gg}}]\cdot\mathbf{1}_n \tag{A1a}$$

or using equation (8b) to replace the negative member in (A1a) there results

$$\mathbf{\Sigma} = (1-\omega_0)4K_t\mathbf{VI}\cdot\mathbf{1}_n + \omega_0\overline{\mathbf{gs}}\cdot\mathbf{1}_m. \tag{A1b}$$

Equation (A1b) with the aid of (C3) leads to $\mathbf{\Sigma} > \mathbf{0}$ for all $0 \leqslant \omega_0 \leqslant 1$ and $K_t > 0$. Thus $\mathbf{P}^{-1}$ is irreducibly diagonally dominant and $\mathbf{P}$ exists for $\omega_0 \neq 0$ and $K_t > 0$ [6]. Moreover, since $\mathbf{\Sigma} > \mathbf{0}$ guarantees the diagonal elements of $\mathbf{P}^{-1}$ to be positive, inverse matrix $\mathbf{P}$ is then itself positive for $\omega_0 \neq 0$ and $K_t > 0$ [6]. This fact, with aid of (C3), also leads to $\mathbf{L} > \mathbf{0}$ for $K_t > 0$ and *all* $0 \leqslant \omega_0 \leqslant 1$.

Multiple reflection matrix $\mathbf{R}$

Here it suffices to examine the existence of $\mathbf{T}$ in equation (22). Note that $\mathbf{L} > \mathbf{0}$ guarantees the off-diagonal elements of $\mathbf{T}^{-1}$ to be non-positive. Now the m-vector of row sums, $\mathbf{\Sigma}$, of $\mathbf{T}^{-1}$ is given by

$$\mathbf{\Sigma} = [(\mathbf{A}/\rho)\mathbf{I}_{2,2} - (\overline{\mathbf{ss}}_{2,2} + \omega_0\mathbf{L}_{2,2})]\cdot\mathbf{1}_m \tag{A2a}$$

and partitioning equations (8) produces

$$\mathbf{AI}_{2,2}\cdot\mathbf{1}_r = \overline{\mathbf{ss}}_{2,1}\cdot\mathbf{1}_b + \overline{\mathbf{ss}}_{2,2}\cdot\mathbf{1}_r + \overline{\mathbf{sg}}_2\cdot\mathbf{1}_n \tag{A3a}$$

$$4K_t\mathbf{VI}\cdot\mathbf{1}_n = \overline{\mathbf{gg}}\cdot\mathbf{1}_n + \overline{\mathbf{gs}}_1\cdot\mathbf{1}_b + \overline{\mathbf{gs}}_2\cdot\mathbf{1}_r \tag{A3b}$$

which, when combined with (A2a) yields

$$\mathbf{\Sigma} = (\varepsilon\mathbf{A}/\rho)\mathbf{I}_{2,2}\cdot\mathbf{1}_r + \overline{\mathbf{ss}}_{2,1}\cdot\mathbf{1}_b + \omega_0\mathbf{L}_{2,1}\cdot\mathbf{1}_b + (1-\omega_0)4K_t\overline{\mathbf{sg}}\cdot\mathbf{P}\cdot\mathbf{VI}\cdot\mathbf{1}_n. \tag{A2b}$$

Consideration of (A2b), with aid of (C3), and $\mathbf{L} > \mathbf{0}$, leads to $\mathbf{\Sigma} > \mathbf{0}$ for all $0 \leqslant \omega_0 \leqslant 1$ and $K_t > 0$. For $K_t > 0$, $\mathbf{T}^{-1}$ is thus strictly diagonally dominant and $\mathbf{T}$ exists [6].

It remains to demonstrate the existence of $\mathbf{T}$ for $K_t = 0$. Now if the enclosure contains at least one black zone, $\overline{\mathbf{ss}}_{2,2}$ may be reducible and we must note that the grey zones can be renumbered so as to partition $\overline{\mathbf{ss}}$ more generally into *normal* form, viz.

$$\overline{\mathbf{ss}} = \left[\begin{array}{c|c|c|c|c} \overline{\mathbf{ss}}_{1,1} & \overline{\mathbf{ss}}_{1,2}^{(1)} & \overline{\mathbf{ss}}_{1,2}^{(2)} & & \overline{\mathbf{ss}}_{1,2}^{(p)} \\ \hline \overline{\mathbf{ss}}_{2,1}^{(1)} & \overline{\mathbf{ss}}_{2,2}^{(1)} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \overline{\mathbf{ss}}_{2,1}^{(2)} & \mathbf{0} & \overline{\mathbf{ss}}_{2,2}^{(2)} & \mathbf{0} & \mathbf{0} \\ \hline \vdots & \mathbf{0} & \mathbf{0} & \searrow & \mathbf{0} \\ \hline \overline{\mathbf{ss}}_{2,1}^{(p)} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\mathbf{ss}}_{2,2}^{(p)} \end{array}\right] \quad \text{(A4)}$$

In (A4) each of the $\overline{\mathbf{ss}}_{2,2}^{(k)}$ sub-matrices is square and either irreducible or the (1×1) null matrix. Further, from (C1) each $\overline{\mathbf{ss}}_{2,1}^{(k)}$ must contain at least one positive entry. Matrix $\mathbf{T}^{-1}$ will then be block diagonal where the row sums of each of the p blocks are found from (A2b) as

$$\mathbf{\Sigma}^{(k)} = (\varepsilon \mathbf{A}/\rho)\mathbf{I}_{2,2}^{(k)} \cdot \mathbf{1}_{kr} + \overline{\mathbf{ss}}_{2,1}^{(k)} \cdot \mathbf{1}_b \qquad k = 1, 2, \ldots, p \quad \text{(A5)}$$

where kr is the dimension of $\overline{\mathbf{ss}}_{2,2}^{(k)}$ and $\mathbf{\Sigma}^{(k)}$ is a kr-vector. Consideration of (A5) with aid of (C2) yields the conclusions that when $K_t = 0$ and the enclosure contains at least one black surface zone, $\mathbf{T}$ will exist and be block diagonal. Moreover, each of the blocks will be positive. If there are no black zones $\overline{\mathbf{ss}}_{2,1}^{(k)}$ will not be defined but $\overline{\mathbf{ss}}_{2,2} \equiv \overline{\mathbf{ss}}$ will be irreducible [after (C1)]. In this instance equation (A5) leads to the requirement that there be at least one $\varepsilon_i \neq 0$ for $\mathbf{T}^{-1}$ to be irreducibly diagonally dominant.

The renumbering of grey zones leading to the partitioning in (A4) can be accomplished with the aid of the directed graph for $\overline{\mathbf{ss}}$ [6]. Here each $\overline{\mathbf{ss}}_{2,2}^{(k)}$ represents an "isolated pocket" of grey surface zones which have no direct view of any other grey zones. This device may be used to practical advantage.

LA METHODE DE ZONAGE: RELATIONS MATRICIELLES EXPLICITES POUR DES SURFACES EN ECHANGE TOTAL

Résumé—On développe des formules matricielles explicites pour le calcul des surfaces en échange total dans le cadre de la méthode de zonage de Hottel. Des relations pratiques sont obtenues pour le cas général d'un milieu gris absorbant et émetteur, isotropiquement dispersif, confiné dans une enceinte grise. L'approche conduit simplement à des cas limites et réduit sensiblement la procédure de calcul. Pour une enceinte découpée en n volumes et r surfaces non noires, la procédure générale requiert l'évaluation d'une matrice $(n \times n)$ et d'une matrice inverse $(r \times r)$. Des conditions suffisantes pour l'existence de la dernière sont montrées absolument non restrictives.

DIE ZONENMETHODE: EXPLIZITE MATRIZENBEZIEHUNGEN FÜR DIE GESAMTAUSTAUSCHFLÄCHEN

Zusammenfassung—Explizite Matrizenbeziehungen werden für die Berechnung der Gesamtaustauschflächen mit der Zonenmethode von Hottel abgeleitet. Es werden für den allgemeinen Fall eines grauen, absorbierend-ausstrahlenden/isotrop-streuenden Mediums, das sich in einem Lambert-Hohlraum befindet, Berechnungsgleichungen gewonnen. Die Lösung führt leicht auf Grenzfälle und reduziert bedeutend die rechnerische Auswertung. Für einen Hohlraum, eingeteilt in n Volumen- und r nicht-schwarze Oberflächenzonen erfordert das allgemeine Verfahren die Auswertung einer $(n \times n)$ und $(r \times r)$ inversen Matrix. Es werden für das Bestehen der letzteren ausreichende Bedingungen gezeigt, die mit keiner Einschränkung verbunden sind.

ЗОНАЛЬНЫЙ МЕТОД. ЯВНЫЕ МАТРИЧНЫЕ СООТНОШЕНИЯ ДЛЯ ОБЩЕЙ ПЛОЩАДИ ОБМЕНА

Аннотация — В приложении к зональному методу Хоттеля выведены явные матричные формулы для расчета общей площади обмена. Получены рабочие соотношения для общего случая серой поглощающе-излучающей изотропно рассеивающей среды в полости Ламберта. Этот метод легко приводит к предельным случаям и значительно облегчает расчеты. Для полости, разделенной на n-объёмных зон и r-нечерных поверхностных зон, общая методика требует определения одной $(n \times n)$ и одной $(r \times r)$ обратных матриц. Показано, что достаточные условия для существования этих матриц являются совсем не ограничительными.